STRICTLY ERGODIC SYMBOLIC DYNAMICAL SYSTEMS

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1. Introduction

We continue the study of strictly ergodic symbolic dynamical systems which was started in our earlier report [6]. The main tools used in this investigation are "homomorphisms" and "substitutions". Among other things, we construct two strictly ergodic symbolic dynamical systems which are weakly mixing but not strongly mixing.

2. Strictly ergodic symbolic dynamical systems

Let A be a finite set consisting of more than one element. Let

$$(2.1) \qquad X = A^Z = \prod_{n \in Z} A_n, \qquad A_n = A \quad \text{for all } n \in Z,$$

be the set of all two sided infinite sequences

$$(2.2) \qquad x = \{a_n \mid n \in Z\}, \qquad a_n = A \quad \text{for all } n \in Z,$$

where

$$(2.3) \qquad Z = \{n \mid n = 0, \pm 1, \pm 2, \cdots\}$$

is the set of all integers. For each $n \in Z$, a_n is called the nth *coordinate* of x, and the mapping

$$(2.4) \qquad \pi_n \colon x \to a_n = \pi_n(x)$$

is called the nth *projection* of the *power space* $X = A^Z$ onto the *base space* $A_n = A$. The space X is a totally disconnected, compact, metrizable space with respect to the usual direct product topology.

Let φ be a one to one mapping of $X = A^Z$ onto itself defined by

$$(2.5) \qquad \pi_n\big(\varphi(x)\big) = \pi_{n+1}(x) \qquad \text{for all } n \in Z.$$

The mapping φ is a homeomorphism of X onto itself and is called the *shift transformation*. The dynamical system (X, φ) thus obtained is called the *shift dynamical system*.

Let X_0 be a nonempty closed subset of X which is invariant under φ. The pair (X_0, φ) may be considered as a dynamical system, and is called the *symbolic dynamical system*. We are interested in the case when (X_0, φ) is *strictly ergodic*, that is, when (X_0, φ) is (i) *minimal*, and (ii) *uniquely ergodic* at the same time. This means that (i) for any $x_0 \in X_0$, the *orbit* of x_0 defined by

$$(2.6) \qquad \mathrm{Orb}(x_0) = \{\varphi^n(x_0) \,|\, n \in Z\}$$

is dense in X_0, and that (ii) there exists only one probability measure defined on the sigma-field $\mathscr{B}_{X_0}$ of all Borel subsets of X_0 which is invariant under φ. Strictly ergodic dynamical systems were discussed by J. C. Oxtoby [9].

Let (X_0, φ) be a strictly ergodic symbolic dynamical system. Let μ_{X_0} be the uniquely determined probability measure defined on the sigma-field $\mathscr{B}_{X_0}$ of all Borel subsets of X_0 which is invariant under φ. The map φ may be considered a measure preserving transformation defined on the probability space $(X_0, \mathscr{B}_{X_0}, \mu_{X_0})$. It is easy to see that φ is *ergodic* as a measure preserving transformation on $(X_0, \mathscr{B}_{X_0}, \mu_{X_0})$, that is, if B is a Borel subset of X_0 such that $\varphi(B) = B$, then either $\mu_{X_0}(B) = 0$ or $\mu_{X_0}(X_0 - B) = 0$.

It is an interesting problem to study the properties of such an ergodic measure preserving transformation φ. All the information concerning the properties of φ as a measure preserving transformation on the probability space $(X_0, \mathscr{B}_{X_0}, \mu_{X_0})$ is contained in the two sided infinite sequence $x_0 = \{a_n \,|\, n \in Z\}$ for any $x_0 \in X_0$. It is well known that, for any $x_0 = \{a_n \,|\, n \in Z\} \in X_0$, the correlation

$$(2.7) \qquad \xi(n) = \lim_{m \to \infty} \frac{1}{m} \sum_{k=\ell}^{\ell+m-1} a_k a_{k+n}$$

exists uniformly in ℓ for any $n \in Z$, and that the correlation function $\{\xi(n) \,|\, n \in Z\}$ plays a fundamental role in the study of the dynamical system (X_0, φ). It turns out that the triple correlation

$$(2.8) \qquad \xi(n_1, n_2) = \lim_{m \to \infty} \frac{1}{m} \sum_{k=\ell}^{\ell+m-1} a_k a_{k+n_1} a_{k+n_2}$$

also exists uniformly in ℓ for any $n_1, n_2 \in Z$, and these correlations $\xi(n)$, $\xi(n_1, n_2)$, together determine the ergodic measure preserving transformation φ up to a spatial isomorphism.

The general theory of symbolic dynamical systems was developed by G. A. Hedlund and M. Morse [2] and W. Gottschalk and G. A. Hedlund [1]. Various examples of strictly ergodic symbolic dynamical systems were discussed by S. Kakutani [6], M. Keane [7], K. Jacobs and M. Keane [4], K. Jacobs [3], and it was shown that many different types of point and continuous spectra can appear in this way.

It was only recently that R. Jewett [5] succeeded in proving that, for any weakly mixing ergodic measure preserving transformation φ' (defined on a Lebesgue probability space) with finite entropy, there exists a strictly ergodic

symbolic dynamical system (X_0, φ) such that the shift transformation φ on the probability space $(X_0, \mathscr{B}_{X_0}, \mu_{X_0})$ is spatially isomorphic with the given transformation φ'. This result was further improved by W. Krieger [8] by showing that the assumption of weak mixing is not necessary. W. Krieger even showed that the finite set A can be chosen in such a way that $e^h < |A| \leq e^h + 1$, where $|A|$ denotes the number of elements in A and $h = h(\varphi')$ is the entropy of φ'.

3. Example 1: Sturmian systems

Let A be a finite set consisting of two elements $+1$ and -1. Let α, β be two real numbers, $0 < \alpha, \beta < 1$. We assume that α is an irrational number. Let f_β, f_β^* be two A valued functions defined on the real line R, periodic with period 1, defined by

$$(3.1) \qquad f_\beta(s) = \begin{cases} +1 & \text{if} \quad 0 \leq s < \beta, \\ -1 & \text{if} \quad \beta \leq s < 1, \end{cases}$$

$$(3.1^*) \qquad f_\beta^*(s) = \begin{cases} +1 & \text{if} \quad 0 < s \leq \beta, \\ -1 & \text{if} \quad \beta < s \leq 1. \end{cases}$$

We may consider f_β and f_β^* as A valued functions defined on the set T of all real numbers mod 1. Let us put

$$(3.2) \qquad x_0 = \{a_n \mid n \in Z\},$$

where

$$(3.3) \qquad a_n = f_\beta(n\alpha) \qquad \text{for all } n \in Z,$$

and $X_0 = \overline{\text{Orb}(x_0)}$ (the closure of the orbit of x_0). Then it is easy to see that (X_0, φ) is strictly ergodic and that φ has a pure point spectrum as an ergodic measure preserving transformation defined on the probability space $(X_0, \mathscr{B}_{X_0}, \mu_{X_0})$. In fact, it is not difficult to see that X_0 consists of all elements x_s and x_s^* of $X = A^Z$ of the form

$$(3.4) \qquad x_s = \{a_n(s) \mid n \in Z\} \qquad \text{for all } s \in T,$$

$$(3.4^*) \qquad x_s^* = \{a_n^*(s) \mid n \in Z\} \qquad \text{for all } s \in T,$$

where

$$(3.5) \qquad a_n(s) = f_\beta(s + n\alpha), \qquad\qquad n \in Z,$$

$$(3.5^*) \qquad a_n^*(s) = f_\beta^*(s + n\alpha), \qquad\qquad n \in Z,$$

It should be observed that x_s and x_s^* differ from each other only for a countable number of values of s, that is, only for those values of $s \in T$ for which $s + n\alpha \equiv 0$ or $s + n\alpha \equiv \beta \pmod{1}$ for some $n \in Z$. If we put

$$(3.6) \qquad \psi(x_s) = \psi(x_s^*) = s, \qquad\qquad s \in T,$$

then ψ is a continuous homomorphism of the dynamical system (X_0, φ) onto the dynamical system (T, φ'), where $\varphi'(s) \equiv s + \alpha \pmod 1$ for any $s \in T$. Since ψ is essentially a one to one mapping, it follows that φ is spatially isomorphic to φ', and, since α is an irrational number by assumption, φ has a pure point spectrum as an ergodic measure preserving transformation defined on the probability space $(X_0, \mathscr{B}_{X_0}, \mu_{X_0})$.

Now, consider $x_0 = \{a_n \mid n \in Z\}$ defined by (3.3) as a two sided infinite sequence

$$(3.7) \qquad x_0 = \{\cdots, a_{-2}, a_{-1}, a_0, a_1, a_2, \cdots\}.$$

If we substitute two successive $+1$ for each $a_n = +1$ in this sequence while keeping each $a_n = -1$ unchanged, then we obtain a new two sided infinite sequence

$$(3.8) \qquad y_0 = \{\cdots, b_{-2}, b_{-1}, b_0, b_1, b_2, \cdots\}.$$

To be more precise, we first define the two sided infinite sequence of integers $\{u_k \mid k \in Z\}$ by

$$(3.9) \qquad \begin{cases} u_0 = 0, \\ u_{k+1} - u_k = \begin{cases} 2 & \text{if } a_k = +1, \\ 1 & \text{if } a_k = -1, \ k = \pm 1, \ \pm 2, \cdots. \end{cases} \end{cases}$$

Then we put

$$(3.10) \qquad \begin{cases} b_{u_k} = b_{u_k+1} = 1 & \text{if } a_k = +1, \\ b_{u_k} = -1 & \text{if } a_k = -1, \ k \in Z. \end{cases}$$

This determines the two sided infinite sequence $\{b_n \mid n \in Z\}$ uniquely.

Let $Y_0 = \overline{\text{Orb}(y_0)}$. Then it is easy to see that (Y_0, φ) is strictly ergodic, although it is not easy to determine the spectrum of φ as an ergodic measure preserving transformation on the probability space $(Y_0, \mathscr{B}_{Y_0}, \mu_{Y_0})$. We mention one result: if α is a transcendental number of Liouville type defined by

$$(3.11) \qquad \alpha = \sum_{k=1}^{\infty} 10^{-n_k},$$

where $\{n_k \mid k = 1, 2, \cdots\}$ is an increasing sequence of positive integers such that $\lim_{k \to \infty} (n_{k+1} - 2n_k) = +\infty$, and if β is a real number for which the fractional part of $10^{n_k}\beta$ is between 0.5 and 0.6 for $k = 1, 2, \cdots$, then (Y_0, φ) is a strictly ergodic dynamical system for which φ is weakly mixing but not strongly mixing as an ergodic measure preserving transformation defined on the probability space $(Y_0, \mathscr{B}_{X_0}, \mu_{X_0})$.

4. Example 2: Morse and Toeplitz sequences

Let again A be the set consisting of two elements $+1$ and -1. Let $\rho(n)$ be an A valued function defined for $n = 0, 1, 2, \cdots$ by

(4.1)
$$\rho(n) = (-1)^{\eta_1 + \eta_2 + \cdots + \eta_k},$$

where $\eta_i = 0$ or 1, $i = 1, 2, \cdots, k$, and

(4.2)
$$n = \eta_1 + \eta_2 2 + \eta_3 2^2 + \cdots \eta_k 2^{k-1}.$$

It is easy to see that $\rho(n)$ is uniquely determined for $n = 0, 1, 2, \cdots$, by $\rho(0) = 1$ and

(4.3)
$$\rho(2n) = \rho(n), \qquad \rho(2n + 1) = -\rho(n), \qquad n = 0, 1, 2, \cdots.$$

Let us consider the element $x_0 = \{a_n \mid n \in Z\} \in A^Z$ defined by

(4.4)
$$a_n = \begin{cases} \rho(n), & n = 0, 1, 2, \cdots, \\ \rho(-n - 1), & n = -1, -2, \cdots, \end{cases}$$

and put $X_0 = \overline{\mathrm{Orb}(x_0)}$. Then $x_0 = \{a_n \mid n \in Z\}$ is the so-called Morse sequence, and, as was observed in [6], the symbolic dynamical system (X_0, φ) is strictly ergodic.

Let now ψ be a mapping of $X = A^Z$ onto itself defined by

(4.5)
$$\pi_n(\psi(x)) = \pi_{n-1}(x)\pi_n(x) \qquad \text{for all } n \in Z.$$

It is easy to see that ψ is a continuous mapping of X onto itself and satisfies $\varphi(\psi(x)) = \psi(\varphi(x))$ for all $x \in X$. This means that ψ is a continuous homomorphism of (X, φ) onto itself. It is also easy to see that ψ is a two to one mapping of X onto itself and that $\psi(x) = \psi(x')$ if and only if $\tau(x) = x'$, where τ is a homeomorphism of X onto itself of period 2 defined by

(4.6)
$$\pi_n(\tau(x)) = -\pi_n(x) \quad \text{for all } n \in Z.$$

Let us put $y_0 = \psi(x_0)$, or equivalently $y_0 = \{b_n \mid n \in Z\}$, where $b_n = a_{n-1}a_n$ for all $n \in Z$. It is easy to see that

(4.7)
$$b_n = \begin{cases} +1 & \text{if } n = 0, \\ -1 & \text{if } n \text{ is odd}, \\ (-1)^{k+1} & \text{if } n \text{ is divisible by } 2^k, \text{ but not divisible by } 2^{k+1}, \\ & \hspace{6em} k = 1, 2, \cdots. \end{cases}$$

This shows that $y_0 = \{b_n \mid n \in Z\}$ is a two sided infinite sequence of Toeplitz type discussed by K. Jacobs and M. Keane [4].

As was shown by Jacobs and Keane [4], if we put $Y_0 = \overline{\mathrm{Orb}(y_0)}$, then (Y_0, φ) is a strictly ergodic dynamical system and φ has a pure point spectrum as an ergodic measure preserving transformation defined on the probability space $(Y_0, \mathscr{B}_{Y_0}, \mu_{Y_0})$.

We denote by $\mathscr{H}_{X_0}$ and $\mathscr{H}_{Y_0}$ the complex L^2 spaces over the probability spaces $(X_0, \mathscr{B}_{X_0}, \mu_{X_0})$ and $(Y_0, \mathscr{B}_{Y_0}, \mu_{Y_0})$, respectively. We also denote by V_φ the unitary operator defined on $\mathscr{H}_{X_0}$ (and $\mathscr{H}_{Y_0}$) by $(V_\varphi f)(x) = f(\varphi(x))$. (We use the same notation V_φ because there is no danger of confusion). Let $\mathscr{M}_e$ and $\mathscr{M}_0$ be the closed linear subspaces of $\mathscr{H}_{X_0}$ consisting of all functions $f \in \mathscr{H}_{X_0}$ such that $f(\tau(x)) = f(x)$ for all $x \in X_0$ (even functions), and $f(\tau(x)) = -f(x)$ for all $x \in X_0$

(odd functions), respectively. Both $\mathcal{M}_e$ and $\mathcal{M}_0$ are invariant under V_φ, orthogonal to each other, and together span the space $\mathcal{H}_{X_0}: \mathcal{H}_{X_0} = \mathcal{M}_e \oplus \mathcal{M}_0$. We now observe that $Y_0 = \psi(X_0)$ and that ψ is a continuous homomorphism of (X_0, φ) onto (Y_0, φ). From the fact observed above that $\psi(x) = \psi(x')$ if and only if $x' = \tau(x)$, it follows that V_φ on $\mathcal{M}_e$ is spectrally isomorphic with V_φ on $\mathcal{H}_{Y_0}$. This shows that V_φ has a pure point spectrum on $\mathcal{M}_e$.

In order to prove that V_φ has a continuous singular spectrum on $\mathcal{M}_0$, we first observe that the function π_0 (projection to the 0th coordinate) is an odd function and that

$$(4.8) \qquad (V^n \pi_0, \pi_0) = (\pi_n, \pi_0) = \int_{X_0} \pi_n(x) \pi_0(x) \mu_{X_0}(dx)$$

$$= \lim_{m \to \infty} \frac{1}{m} \sum_{k=0}^{m-1} \pi_n(\varphi^k(x_0)) \pi_0(\varphi^k(x_0))$$

$$= \lim_{m \to \infty} \frac{1}{m} \sum_{k=0}^{m-1} \pi_{n+k}(x_0) \pi_k(\pi_0)$$

$$= \lim_{m \to \infty} \frac{1}{m} \sum_{k=0}^{m-1} \rho(n+k)\rho(k).$$

If we denote this limit by $\sigma(n)$, then it is easy to see that σ is a positive definite function defined on Z which satisfies the following conditions:

$$(4.9) \qquad \begin{array}{l} \sigma(0) = 1, \qquad \sigma(-n) = \sigma(n), \\ \sigma(2n) = \sigma(n), \\ \sigma(2n+1) = -\tfrac{1}{2}(\sigma(n) + \sigma(n+1)), \qquad n = 0, 1, 2, \cdots. \end{array}$$

Let $v(\lambda)$ be a real valued, nondecreasing function defined on the unit interval $[0, 1]$, continuous on the right at every point, such that

$$(4.10) \qquad \sigma(n) = \int_0^1 \exp\{2n\pi i\lambda\}\, dv(\lambda) \qquad \text{for all } n \in Z.$$

From (4.9) follows that

$$(4.11) \qquad \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} (\sigma(k))^2 = 0,$$

and this shows that $v(\lambda)$ is a continuous function. On the other hand, from the second and third rows of (4.9), it follows that

$$(4.12) \qquad dv\left(\frac{\lambda}{2}\right) + dv\left(\frac{\lambda+1}{2}\right) = dv(\lambda),$$

$$(4.13) \qquad dv\left(\frac{\lambda}{2}\right) - dv\left(\frac{\lambda+1}{2}\right) = -\cos \pi\lambda\, dv(\lambda)$$

for all λ, and hence

$$(4.14) \qquad \frac{1}{2}\left(v'\left(\frac{\lambda}{2}\right) + v'\left(\frac{\lambda + 1}{2}\right)\right) = v'(\lambda),$$

$$(4.15) \qquad \frac{1}{2}\left(v'\left(\frac{\lambda}{2}\right) - v'\left(\frac{\lambda + 1}{2}\right)\right) = -\cos \pi\lambda v'(\lambda),$$

for almost all λ, where $v'(\lambda)$ denotes the derivative of $v(\lambda)$ which exists almost everywhere. We observe that $v'(\lambda)$ is integrable on the unit interval $[0, 1]$, and if we denote by $\gamma(n)$ the nth Fourier coefficient of $v'(\lambda)$:

$$(4.16) \qquad \gamma(n) = \int_0^1 v'(\lambda) \exp\{2n\pi i\lambda\}\, d\lambda,$$

then from (4.14) follows that $\gamma(2n) = \gamma(n)$ for all $n \in Z$. Since $\lim_{n \to \pm \infty} \gamma(n) = 0$ by the Riemann-Lebesgue theorem, we must have $\gamma(n) = 0$ for all $n \neq 0$, and hence $v'(\lambda) = $ constant almost everywhere. This constant must be 0 because of (4.15). This shows that $v(\lambda)$ is singular.

Let $f \in \mathcal{M}_0$ be an odd function of the form $f = \pi_0 \cdot g$, where g is a normalized eigenfunction from $\mathcal{M}_e$ belonging to the eigenvalue λ_0: $V_\varphi g = \exp\{2\pi i\lambda_0\}\, g$. Since φ is ergodic on $(X_0, \mathcal{B}_{X_0}, \mu_{X_0})$, we have $|g(x)| = 1$ almost everywhere on X_0. We observe that finite linear combinations of such functions f form a dense subset of $\mathcal{M}_0$. (This follows from the fact that V_φ has a pure point spectrum on $\mathcal{M}_e$.) Hence, in order to show that V_φ has a continuous singular spectrum on $\mathcal{M}_0$, it suffices to show that each such function f has a continuous singular spectrum for V_φ, that is, that if $v_f(\lambda)$ is a real valued, nondecreasing function defined on the unit interval $[0, 1]$ such that $(V_\varphi^n f, f) = \int_0^1 \exp\{2n\pi i\lambda\}\, dv_f(\lambda)$ for all $n \in Z$, then $v_f(\lambda)$ is continuous and singular. This is, however, easy to verify since

$$(4.17) \qquad (V_\varphi^n f, f) = \int_{X_0} \pi_n(x)\pi_0(x) \exp\{2n\pi i\lambda\}\, g(x)\overline{g(x)}\mu_{X_0}(dx)$$

$$= \exp\{2n\pi i\lambda_0\}\sigma(n) = \int_0^1 \exp\{2n\pi i(\lambda + \lambda_0)\}\, dv(\lambda).$$

5. Example 2 continued

Let $y_0 = \{b_n \mid n \in Z\}$ be the two sided infinite sequence of Toeplitz type defined by (4.7). We construct a two sided infinite sequence $z_0 = \{c_n \mid n \in Z\}$ from $y_0 = \{b_n \mid n \in Z\}$ in exactly the same way as we obtained the sequence $y_0 = \{b_n \mid n \in Z\}$ from $x_0 = \{a_n \mid n \in Z\}$ in the discussion of Example 1 (that is, by substituting for each $a_n = +1$ two successive $+1$, while keeping each $a_n = -1$ unchanged), and consider the orbit closure $Z_0 = \overline{\mathrm{Orb}(z_0)}$. Then it is again easy to see that (Z_0, φ) is strictly ergodic, although it is not easy to calculate the spectrum of φ as a measure preserving transformation on the probability space $(Z_0, \mathcal{B}_{Z_0}, \mu_{Z_0})$. In our case, it is again possible to show that φ is weakly mixing but not strongly mixing on $(Z_0, B_{Z_0}, \mu_{Z_0})$.

REFERENCES

[1] W. H. GOTTSCHALK and G. A. HEDLUND, *Topological Dynamics*, American Mathematical Society Colloquium Publications, No. 36, 1955.

[2] G. A. HEDLUND and M. MORSE, "Symbolic dynamics," *Amer. J. Math.*, Vol. 60 (1938), pp. 815–866; Vol. 62 (1940), pp. 1–42.

[3] K. JACOBS, "Maschinenerzeugte 0-1-Folgen," *Selecta Mathematica I*, Heidelberger Taschenbücher, No. 49, 1969, pp. 1–27.

[4] K. JACOBS and M. KEANE, "0-1 sequences of Toeplitz type," *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete.*, Vol. 13 (1969), pp. 123–131.

[5] R. I. JEWETT, "The prevalency of uniquely ergodic systems," *J. Math. Mech.*, Vol. 19 (1969–1970), pp. 717–729.

[6] S. KAKUTANI, "Ergodic theory of shift transformations," *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley and Los Angeles, University of California Press, 1967, Vol. 2, Part 2, pp. 405–414.

[7] M. KEANE, "Generalized Morse sequences," *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete.*, Vol. 10 (1968), pp. 335–353.

[8] W. KRIEGER, "On unique ergodicity," *Sixth Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley and Los Angeles, University of California Press, 1972, Vol. 2, pp. 327–346.

[9] J. C. OXTOBY, Ergodic sets, *Bull. Amer. Math. Soc.*, Vol. 58 (1952), pp. 116–136.